Numerical solutions of the variational equations for sandpile dynamics

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A variational form of the energy conservation equation is applied to derive a system of variational inequalities describing the dynamics of sandpile growth on an arbitrary rigid support surface due to an external source of granular material. Water transport through the river networks and formation of lakes is also investigated as a limiting case of the sandpile evolution problem. A numerical algorithm is suggested for the solution of the obtained system of variational inequalities. The developed numerical procedure is applied for the investigation of the pile growth on a number of rigid support surfaces, water transport, and formation of lakes. [S1063-651X(97)12005-0]

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I. INTRODUCTION

The dynamics of sandpiles were studied quite extensively in the past, both theoretically and experimentally (see, e.g., Refs. [1–3]). Most of these studies investigate the details of the pile surface relaxation near a stable configuration by avalanches. Less attention was paid to the more complicated problem of determining the evolution of the mean surface of a pile growing on an arbitrary rigid support surface due to an external source of granular material. This problem arises in numerous technological applications and naturally occurring phenomena, e.g., bulk solids handling, geomorphology, etc.

The first successful continuum model of pile formation that provides quantitative results was proposed by Prigozhin [4,5]. In these studies the inertial effects were neglected; i.e., the system was driven by gravitational and frictional forces only. The mathematical model of heap evolution was proved to be a dual formulation of a time-dependent quasivariational inequality.

In spite of its elegance and mathematical irreproachability this method has a limited range of applications. The inequality used in [4,5] is a variational one only if the support surface of the pile is inclined at an angle less than the angle of repose everywhere. In the opposite case, this inequality becomes a quasivariational inequality, which requires time consuming iterative treatment.

In [6] we suggested a more general variational description of sandpile evolution based on the Hamilton principle of the stationary action. The application of the principle of the stationary action allows one to derive a system of the variational inequalities describing sandpile evolution. In the present investigation we developed a numerical procedure for the solution of these variational inequalities. The developed procedure is applied for the investigation of the pile growth on a number of rigid support surfaces. Water transport through the river networks and formation of lakes is also investigated as a limiting case of the sandpile evolution problem.

II. A MODEL FOR SANDPILE GROWTH ON AN ARBITRARY RIGID SUPPORT SURFACE

Generally the problem of the sandpile dynamics is formulated as follows. Let a cohesionless granular material with bulk density ρ be poured down onto a rigid support surface with a profile $z=h_0(x,y)$, where $(x,y)\in\Omega$ and forms a heap with a free boundary $z=h(x,y)$. All the material lie above the support surface, i.e.,

$$h\geqslant h_0, \tag{1}$$

The mass of the material that falls on the area $d\Omega$ during time interval dt is $\rho w(t,x,y)dt\ d\Omega$, where $\rho w(t,x,y)$ is the surface density of the external source of granular material. The problem is to determine the time dependence of the height of a granular pile $h(t,x,y)$.

Assume that the intensity of the source is small. Then the flow of the granular material occurs only in a thin boundary layer and does not involve the stationary bulk of the material [2]. Denote the horizontal projection of the mass flux density per unit area by $\rho\overline{q}$ and the horizontal component of the integral mass flux density per unit area by $\rho\overline{Q}$, i.e., $\overline{q}=\partial\overline{Q}/\partial t$.

In the present work we consider the simplest model of the ideal incompressible material with Coulomb friction law [7]. It can be argued that such a restrictive assumption ignores most of the essential properties of the real granular materials [1–3]. However, the concept of the ideal material with Coulomb friction has been extensively used in engineering for almost two centuries. It plays the same fundamental role in the mechanics of granular media as the Hook's law in solid mechanics. Therefore we believe that the analysis of a pile of granular material using the Coulomb friction law can form a basis for more general theory.

Since the bulk density is a constant, the mass conservation law can be written as follows:

$$\frac{\partial h}{\partial t}=-\vec{\nabla}\cdot\vec{q}+w. \tag{2}$$

Assume that the domain Ω is bounded by the impermeable boundary Γ, i.e.,

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$$q_n|_\Gamma = 0. \tag{3}$$

Note that all the derivations are valid also for the completely permeable boundary, i.e., for outflow boundary conditions:

$$h(t,x,y)|_\Gamma = h_0|_\Gamma .$$

Since the flow rate of the material is assumed to be small, one can neglect the inertia of the grains and suppose that the system is driven by gravitational and frictional forces only. The expression for the surface density of the potential energy of the sandpile reads as follows:

$$U = \rho g \int_0^h z\ dz = \frac{\rho g}{2} h^2,$$

where g is the acceleration of gravity.

Assume that the equations for the surface density of the dissipation rate of energy in the pile read as follows:

$$\varepsilon = \rho \vec{F} \cdot \vec{q},$$

where the expression for the friction force per unit mass of a flowing granular layer $\vec{F}$ will be specified below.

III. VARIATIONAL FORMULATION FOR SANDPILE DYNAMICS

Using the synchronous variations of the pile height δh and of the horizontal component of the integral flux $\delta \vec{Q}$, the variational form of the energy conservation law can be written as follows (for details see [6]):

$$\rho g \int_\Omega h\ \delta h\ d\Omega = -\rho \int_\Omega \vec{F} \cdot \delta \vec{Q}\ d\Omega. \tag{4}$$

It is clear that the variations δh and $\delta \vec{Q}$ are not independent since the virtual material flux affects the free boundary of the pile. Rewriting the law of mass conservation with respect to the variations we obtain

$$\delta h = -\vec{\nabla} \cdot \delta \vec{Q} \tag{5}$$

Substituting Eq. (5) into Eq. (4) and using Gauss theorem we arrive at the following equation of forces balance:

$$g \vec{\nabla} h = -\vec{F}.$$

Determining the friction force $\vec{F}$ requires a detailed description of the friction mechanisms in a granular material. Such mechanisms were discussed in [8,9], and it was confirmed that forces acting on a layer of sand sliding down a rough wedge correspond to the Coulomb-like friction; i.e., $\vec{F} \propto \vec{u}/|u|$, where $\vec{u}$ is the velocity of granular material.

On the other hand, if the inertia of the granular flow can be neglected, it is reasonable to assume that the granular material slides in the direction of the steepest descent at the pile surface:

$$\vec{q} \propto -\vec{\nabla} h \Rightarrow \vec{F} \propto \vec{q}.$$

It is known that the sandpile avalanches occur if and only if the pile surface is inclined at an angle equal to the angle of repose α_0:

$$|\vec{\nabla} h| = \tan\ \alpha_0 = \gamma.$$

Therefore it is reasonable to assume that the friction force $\vec{F}$ is determined by the following equation:

$$\vec{F} = \gamma g \frac{\vec{q}}{|q|},$$

which implies that $\vec{\nabla} h = -\gamma \vec{q}/|q|$ and $|\vec{\nabla} h| = \gamma$.

Note that the above equations are valid only if $h > h_0$. The situation when the granular material slides down the support surface, i.e., when $h = h_0$, requires a special treatment. We assume that in this case the granular flow is also parallel to $\vec{\nabla} h$:

$$\vec{q} \cdot \mathbf{e} \cdot \vec{\nabla} h = 0 \tag{6}$$

where components of tensor $\mathbf{e}$ in Cartesian coordinates $e_{12} = -e_{21} = -1,\ e_{11} = e_{22} = 0$.

Suppose also that granular material does not slide at the part of the support surface which is inclined at the angle less then the angle of repose:

$$\vec{q} = 0 \quad \text{for } |\vec{\nabla} h| < \gamma.$$

Since $\vec{q} \propto -\vec{\nabla} h$, the latter condition can be written as follows:

$$\vec{q} \cdot \vec{\nabla} h + \gamma |q| \leqslant 0. \tag{7}$$

It is clear that the constraints (6) and (7) are satisfied automatically when $h > h_0$.

IV. NUMERICAL ALGORITHM

In this section we demonstrate that the time dependent variational equation (4) can be solved by reduction to a sequence of the steady-state minimization problems. We restrict our attention to the case that is continuous with respect to the space variables and is discrete with respect to time.

After discretization in time Eq. (2) and the formula for $\delta \vec{Q}$ at time $k\Delta t$ read as follows:

$$h_k = h_{k-1} + (w - \vec{\nabla} \cdot \vec{q}_k)\Delta t,$$

$$\delta \vec{Q}_k = \delta \vec{q}_k \Delta t. \tag{8}$$

Substitute Eqs. (8) into the variational equation (4). Then after simple manipulations we obtain the following minimization problem at time $k\Delta t$:

$$\int_\Omega \left\{ \frac{\Delta t}{2} (\vec{\nabla} \cdot \vec{q}_k)^2 + \gamma |\vec{q}_k| - (h_{k-1} + w\Delta t)\vec{\nabla} \cdot \vec{q}_k \right\} d\Omega \underset{\vec{q}}{\rightarrow} \min, \tag{9}$$

which is to be solved under the constraints (1), (2), (6), (7), and (8).

It must be noted that the time increment Δt appears in the expression for the functional (9) near the quadratic term. Therefore the time increment Δt multiplies the second order derivative in the Euler equations for the functional (9) [see Eq. (A1), Appendix A]. When $\Delta t=0$ the Euler equation for the functional (9) yields the pile equilibrium condition $|\vec{\nabla} h_{k-1}|=\gamma$. The latter condition implies that the pile is stable at the $(k-1)$th time step. Therefore, in spite of the small multiplier near the high order derivative, the Euler equation is not singular since the small term $\propto\Delta t$ in the functional (9) compensates the deviation from the equilibrium state caused by the arrival of the new portion of granular material with mass $\rho w \Delta t$ per unit surface.

Because of a very large number of variables the use of the classical gradient-type methods for solving the above minimization problem is inappropriate. It was concluded in [10] that a more effective approach is to employ a block-relaxation method that takes into account the particular structure of the problem to be solved. However, an essential difficulty in applying a relaxation method arises from the nondifferentiable term $|q|$. Note that $\vec{q}$ enters into the equations not only through its absolute value $|q|$ but also through its derivatives, i.e., nonlocally. In such a case a relaxation procedure will not converge to the solution of the minimization problem [10]. In order to avoid this problem we introduce an additional variable $\vec{p}$ $(\vec{p}=\vec{q})$, as done in [10] for the investigation of the Bingham fluid flow. Replacing the term $|q|$ by $|p|$ we obtain the following minimization problem:

$$\Phi \triangleq \int_{\Omega}\left\{\frac{\Delta t}{2}(\vec{\nabla}\cdot\vec{q}_k)^2+\gamma|\vec{p}_k| - (h_{k-1}+w\Delta t)\vec{\nabla}\cdot\vec{q}_k d\Omega \right. \underset{\vec{q},\vec{p}}{\rightarrow} \min, \tag{10}$$

$$\phi_1 \triangleq h_k - h_{k-1} - (w-\vec{\nabla}\cdot\vec{q}_k)\Delta t = 0, \tag{11}$$

$$\phi_2 \triangleq \vec{q}_k\cdot\vec{\nabla}h_k+\gamma|p_k|+\zeta_k=0, \tag{12}$$

$$\phi_3 \triangleq \vec{q}_k\cdot\mathbf{e}\cdot\vec{\nabla}h_k=0, \tag{13}$$

$$\vec{\phi}_4 \triangleq \vec{q}_k-\vec{p}_k=0, \tag{14}$$

$$\vec{n}\cdot\vec{q}_k|_{\Gamma}=0, \tag{15}$$

$$h_k \geqslant h_0, \tag{16}$$

$$\zeta_k \geqslant 0, \tag{17}$$

where ζ is a slack variable, and $\vec{n}$ is a normal unit vector to the boundary Γ. Hereafter the index k, which denotes the current time, will be omitted in order to simplify notations.

Now the functional (10) and the constraints (11)–(14) are differentiable with respect to the vector $\vec{q}$ while the new variable $\vec{p}$ enters into the above equations locally. Therefore a block-relaxation method can be applied to solve the minimization problem (10)–(17).

In this study we used the method of an augmented Lagrangian [10], which is a combination of the duality and penalty methods. Let us introduce the following Lagrangian multipliers λ, μ_1, μ_2, and $\vec{\nu}$, which correspond to the constraints (11)–(14), respectively. Then the following augmented Lagrangian can be associated with the minimization problem (10)–(17):

$$\begin{aligned} L \triangleq & \Phi + \int_{\Omega} \lambda\phi_1 + r_1\phi_1^2+\mu_1\phi_2+r_2\phi_2^2+\mu_2\phi_3+r_2\phi_3^2+\vec{\nu}\cdot\vec{\phi}_4+r_3\phi_4^2 d\Omega \\ = & \int_{\Omega}\Delta t(1/2+r_1)(\vec{\nabla}\cdot\vec{q})^2+[2r_1h-(1+2r_1)(h_{k-1}+w\Delta t)+\lambda]\vec{\nabla}\cdot\vec{q}+r_3|\vec{q}-\vec{p}|^2+\vec{\nu}\cdot(\vec{q}-\vec{p}) \\ & +\gamma|p|+r_2[(\vec{q}\cdot\vec{\nabla}h+\gamma|p|+\zeta)^2+(\vec{q}\cdot\mathbf{e}\vec{\nabla}h)^2]+\mu_1(\vec{q}\cdot\vec{\nabla}h+\gamma|p|+\zeta)+\mu_2(\vec{q}\cdot\mathbf{e}\cdot\vec{\nabla}h)d\Omega, \end{aligned} \tag{18}$$

where r_i are the penalty parameters.

At each iteration of the augmented Lagrangian algorithm the following problem of minimization is solved:

$$L \xrightarrow[(\vec{q}^{n+1},\vec{p}^{n+1},h^{n+1},\zeta^{n+1})]{} \min,$$

with the constraints (15)–(17).

The Lagrangian multipliers are kept constant. When a minimum of L is attained, the Lagrangian multipliers are recalculated as follows (for details see [10], p. 402):

$$\lambda^{n+1}=\lambda^n+\theta r_1[h-h_{k-1}-(w-\vec{\nabla}\cdot\vec{q})\Delta t],$$

$$\mu_1^{n+1}=\mu_1^n+\theta r_2[\vec{q}\cdot\vec{\nabla}h+\gamma|p|+\zeta], \quad 0<\theta\leqslant 2$$

$$\mu_2^{n+1}=\mu_2^n+\theta r_2(\vec{q}\cdot\mathbf{e}\cdot\vec{\nabla}h),$$

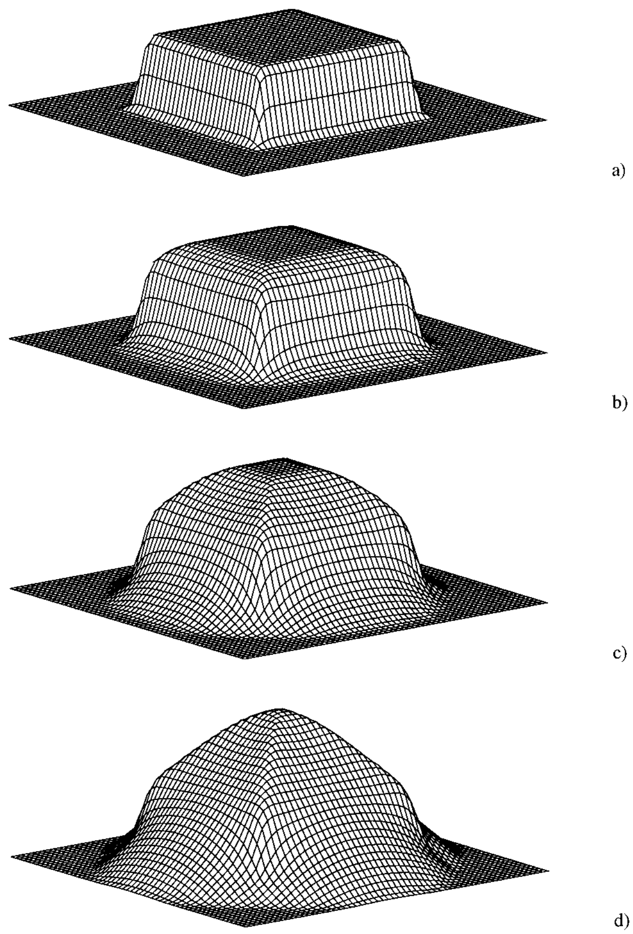

FIG. 1. Pile growth on a support surface (a) from a plane uniform source.

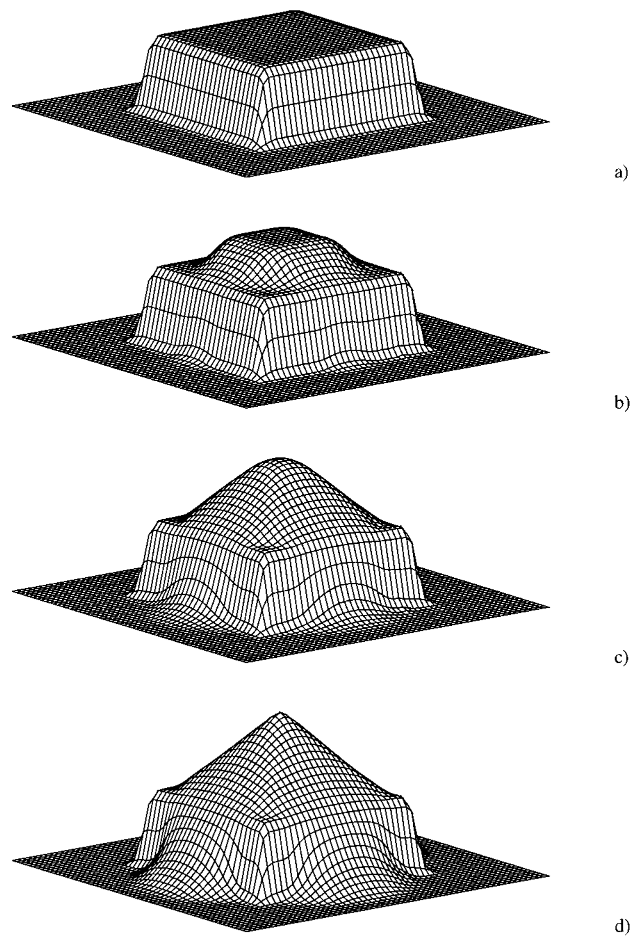

FIG. 2. Pile growth on a support surface (a) from a circle uniform source.

$$\vec{\nu}^{n+1} = \vec{\nu}^{n} + \theta r_3(\vec{q} - \vec{p}),$$

where n is the number of iterations. In numerical calculations the parameter θ was chosen in the interval [0.8,1.2].

As noted above, a minimum of the functional (18) can be determined iteratively by sequentially solving the partial minimization problems with respect to h, $\vec{q}$, $\vec{p}$, and ζ, respectively:

$$\vec{q} = \underset{\vec{q}:\ \vec{q}_n|_\Gamma = 0}{\operatorname{argmin}} L, \tag{19}$$

$$h = \underset{h: h \geqslant h_0}{\operatorname{argmin}} L. \tag{20}$$

When the sandpile height h is determined from the solution of the minimization problem (20) the minimization problem (19) is solved again with the new value of h. This procedure of successive solving minimization problems (19)–(20) is repeated until convergence. Then the one-point minimization problem,

$$\begin{pmatrix} \vec{p} \\ \zeta \end{pmatrix} = \underset{\vec{p},\zeta:\zeta \geqslant 0}{\operatorname{argmin}} L \tag{21}$$

can be solved analytically. The detailed numerical algorithm for the solution of the minimization problems (19)–(21) is described in Appendix A.

The minimization problems (19)–(20) are solved repeatedly with the new values of $\vec{p}$ and ζ, etc., until convergence. Note that the minimization problems (19) and (20) are the problems of quadratic minimization and, therefore, they can be solved using the standard methods (see Appendix A).

The penalty parameters r_i were chosen from the interval [1,10]. At all steps of the numerical procedure the following termination criterion was used:

$$\left(\int_\Omega (\vec{q}^{n} - \vec{q}^{n-1})^2 d\Omega \right)^2 \leqslant 10^{-3}.$$

Notably, when $|\vec{\nabla} h_0| \leqslant \gamma$ everywhere, the slack variable ζ and the constraints (11)–(13) and (16) are not required. Thus the minimization problem can be solved with respect to $\vec{q}$ and after that the sandpile height h can be calculated from the mass conservation equation (8).

V. WATER TRANSPORT

It was noted by Prigozhin [4] that sandpiles and river networks are similar dissipative systems. Let $h_0(x,y)$ be the land surface and $\rho w(t,x,y)$ the intensity of precipitation. As-

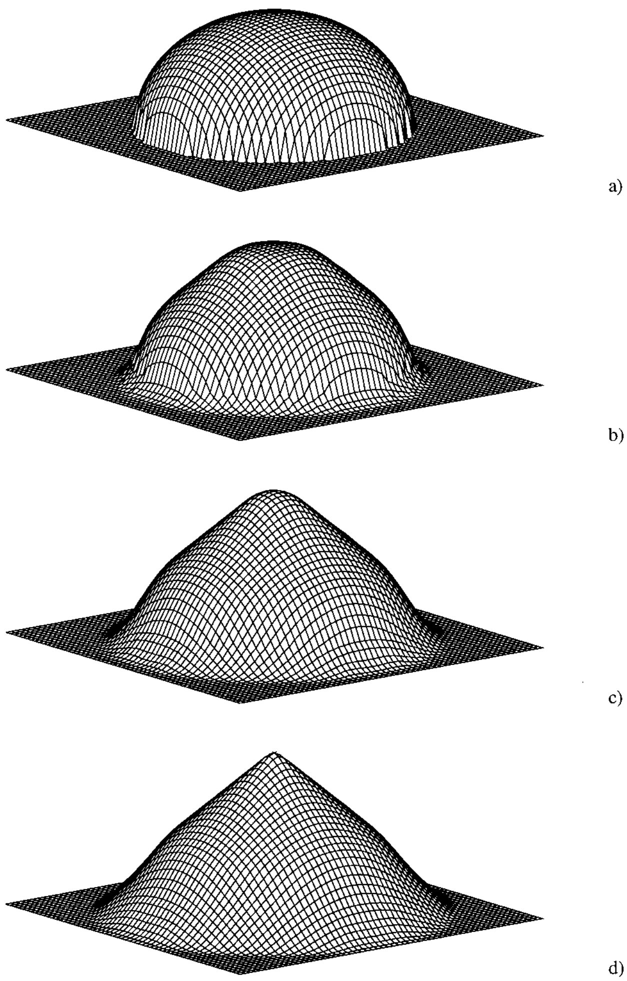

FIG. 3. Pile growth on a support surface (a) from a circle uniform source.

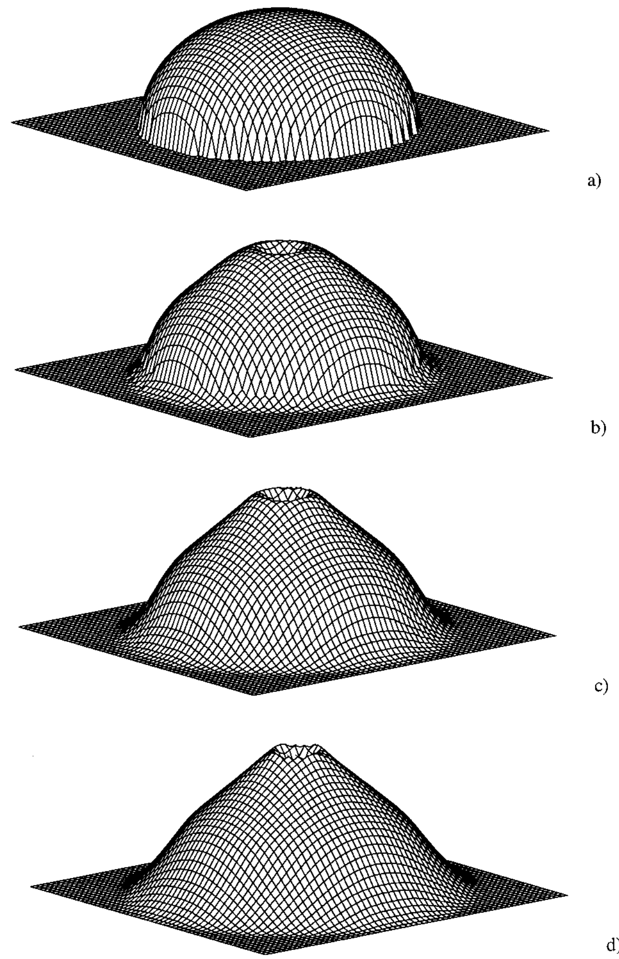

FIG. 4. Pile growth on a support surface (a) from a concentric ringlike uniform source.

sume that water does not evaporate and does not penetrate the soil but just flows down the slopes and forms lakes at local depressions of the land surface. These lakes can be viewed as piles with zero angle of repose. Therefore in the limiting case of the zero γ the model of heap growth on a rigid support surface describes water transport through the river networks and the formation of lakes.

Since when γ is zero the friction forces vanish, the additional variable $\vec{p}$ and the constraints (14) are not needed.

Since the water flow in the lakes is not confined to a thin surface layer only, the assumptions of our model are not valid. However, the flow in lakes does not affect the horizontal water profile. Therefore, in order to specify a lake surface we need only an integral water balance over the whole lake, and a detailed description of the flow in lakes is not required. Note that the solution for a flow field $\vec{q}$ in lakes is invariant with respect to addition of an arbitrary solenoidal field $\tilde{\vec{q}}$. The latter was the reason for the indefinite growth of the magnitudes of the water fluxes, which occurred in some computations. In order to prevent such uncontrollable swirling of water in the lakes the following penalty term was added to the functional (10):

$$\int_{\Omega} r_4(\vec{q}^{\,n}-\vec{q}^{\,n-1})^2 d\Omega,$$

which is the square of the difference between the value of the flux vector at a current iteration and at the previous one. When the minimization problem (10)–(13), (15)–(17) is solved the vector $\vec{q}^{\,n-1}$ is replaced by $\vec{q}^{\,n}$ and the calculations are repeated. The penalty parameter r_4 is chosen in the interval $[0.05–0.1]$ and was sufficient to provide the stability of the numerical algorithm. In the numerical calculations the following modified termination criterion was used:

$$\left(\int_{\Omega/\Omega_1^n}(\vec{q}^{\,n}-\vec{q}^{\,n-1})^2 d\Omega\right)^{1/2}+2\left(\int_{\Omega_1^n}(h^n-h^{n-1})^2 d\Omega\right)^{1/2} \leqslant 10^{-3},$$

where Ω_1^n is the area of the lakes at a current iteration.

VI. NUMERICAL RESULTS AND DISCUSSION

The model developed for the sandpile evolution, water transport, and formation of lakes in the form of the variational equations allows an effective computational realization using the numerical procedure described above.

In order to verify the proposed algorithm we studied sandpile growth on the rigid support surfaces with sharp variations of the profile and with a slope greater than γ. The shapes of growing sandpiles at several times ($t=0$, $\frac{1}{3}\gamma L/w$,

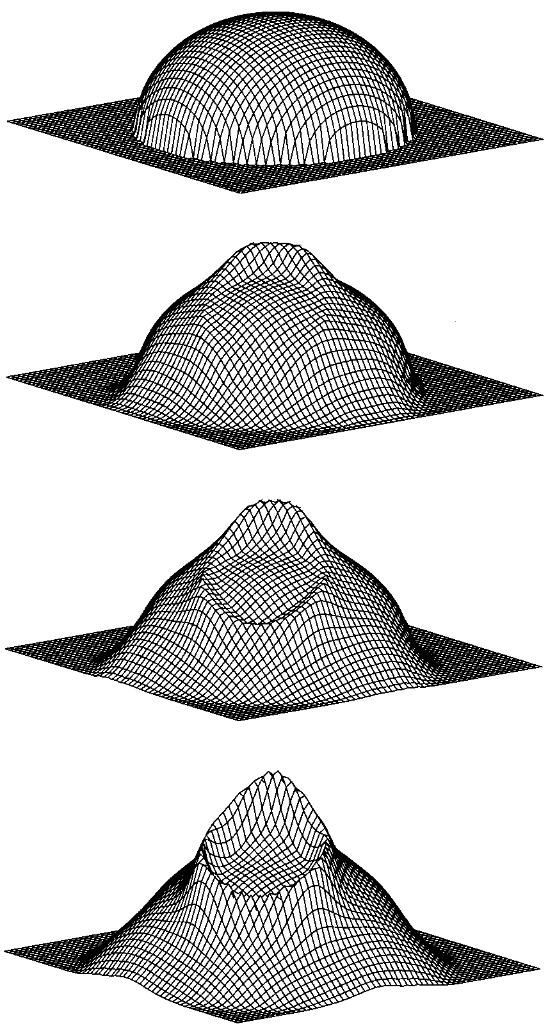

FIG. 5. Pile growth on a support surface (a) from a shifted ringlike uniform source.

$\frac{2}{3}\gamma L/w$, $\gamma L/w$, where L is a characteristic size of the support) are presented in Figs. 1–5. In the calculations L and γ were set equal to unity. Inspection of these figures shows that the shape of the piles depends on the shape of the platform and on the distribution of the density of the source of granular material. The pile is stable until the inclination of the platform is less than the angle of repose of the granular material. When the foot of the pile reaches a location at the support surface with the inclination greater than the repose angle, the granular material slides down the slope and deposits at the regions of the platform that have angles of inclination less than the repose angle. Note that the shape of the pile surface controls only the direction of the material flux, while its magnitude depends on the integral mass and energy balance.

We analyzed also water transport between two communicating reservoirs. The results of the calculations are presented in Fig. 6. A point source of water is located above the right reservoir. The horizontal water profile in the right reservoir rises until water starts to spill into the left reservoir. Thereafter all the water poured into the right reservoir flows into the left reservoir.

Thus it can be concluded that the suggested model for sandpile dynamics allows one to predict a heap growth on an arbitrary rigid support surface under the action of the arbitrary distributed external sources of granular material and formation of lakes at the arbitrary landscapes.

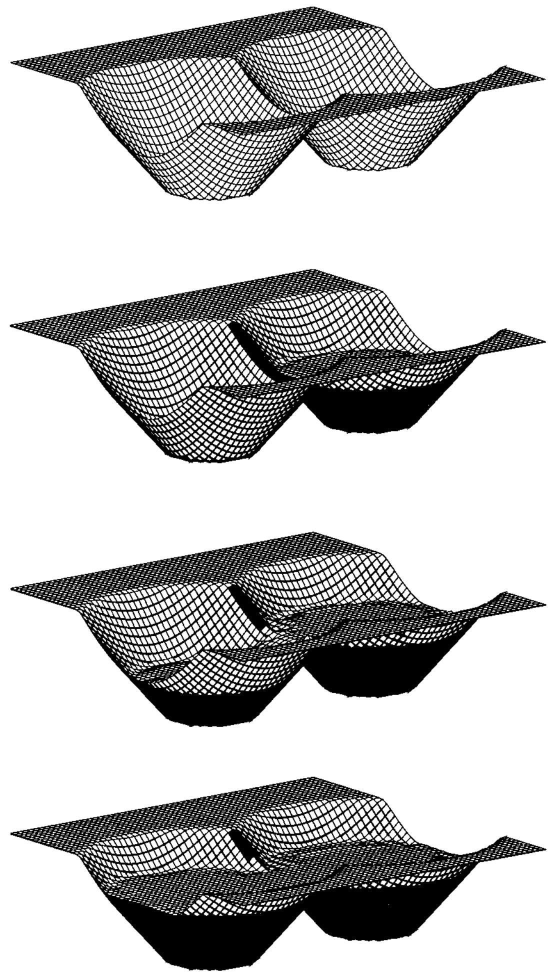

FIG. 6. Water transport through two communicated reservoirs from a point source located above the right reservoir.

APPENDIX: NUMERICAL PROCEDURE FOR MINIMIZATION OF THE AUGMENTED LAGRANGIAN (18)

Here we describe the procedure for minimization of the functional (18) in more detail. The minimization problem (19) yields the following Euler equation, which is linear with respect to $\vec{q}$:

$$\Delta t(1+2r_1)\vec{\nabla}\vec{\nabla}\cdot\vec{q}-2(r_3+r_2|\vec{\nabla}h|^2)\vec{q}$$

$$=\vec{\nabla}[2r_1h-(1+2r_1)(h_{k-1}+w\Delta t)+\lambda]-2r_3\vec{p}-\vec{\nu}$$
$$+(\mu_1+2r_2)(\vec{q}\cdot\vec{\nabla}h+\gamma|p|+\zeta)$$
$$\times\vec{\nabla}h-(\dot{\mu}_2+2r_2)(\vec{q}\cdot\mathbf{e}\cdot\vec{\nabla}h)\vec{\nabla}h\cdot\mathbf{e}$$
$$=0, \quad \text{(A1)}$$

with the fixed values of all other variables and the boundary conditions (15). The above equations were solved by the method of alternating directions with over-relaxation.

The sandpile height h is determined as $\max[\tilde{h},h_0]$, where $\tilde{h}$ is the solution of the following elliptic type equation:

$$2r_1\tilde{h}-\vec{\nabla}\cdot\{\vec{q}[2r_2(\vec{q}\cdot\vec{\nabla}\tilde{h}+\gamma|\vec{p}|+\zeta)+\mu_1]-\mathbf{e}\cdot\vec{q}[2r_2(\vec{q}\cdot\mathbf{e}\cdot\vec{\nabla}\tilde{h})+\mu_2]\}=2r_1(h_{k-1}+w\Delta t-\vec{\nabla}\cdot\vec{q}\Delta t)-\lambda, \quad \text{(A2)}$$

with the boundary conditions $q_\tau(q_\tau\partial\tilde{h}/\partial n+\mu_2)|_r=0$.

The differential equation (A2) is the Euler equation for the minimization of the functional (18) with respect to $\tilde{h}$. In this study Eq. (A2) was solved by the point over-relaxation method. The relaxation parameter ω in solving Eqs. (A1) and (A2) was chosen from the interval [1.2,1.6].

The one-point minimization problems with respect to $\vec{p}$ and ζ can be solved as follows. Euler equations for minimization of the Lagrangian (18) with respect to $\vec{p}$ and ζ read as follows:

$$2r_2(\vec{q}\cdot\vec{\nabla}h+\gamma|p|+\zeta)\,\frac{\vec{p}}{|p|}+\gamma(1+\mu_1)\,\frac{\vec{p}}{|p|}+2r_3\vec{p}=2r_3\vec{q}+\vec{\nu},$$

$$2r_2(\zeta+\vec{q}\cdot\vec{\nabla}h+\gamma|\vec{p}|)+\mu_1=0.$$

The first equation implies that $\vec{p}=p\vec{P}/|P|$, where $\vec{P}=2r_3\vec{q}+\vec{\nu}$.

The above equations can be rewritten as follows:

$$p=\tfrac{1}{2}\{|P|-\gamma[1-\mu_1+2r_2(\vec{q}\cdot\vec{\nabla}h+\zeta)]\}/(r_2\gamma^2+2r_3),$$

$$\zeta=-\vec{q}\cdot\vec{\nabla}h-\gamma p-\mu_1/(2r_2).$$

In order to satisfy the constraints $p\geqslant 0$ and $\zeta\geqslant 0$ these variables must be chosen as follows. If $p\geqslant 0$ and $\zeta\geqslant 0$ the above equations can be solved immediately:

$$p=(|P|-\gamma)/(2r_3),$$

$$\zeta=-\vec{q}\cdot\vec{\nabla}h-\gamma p-\mu_1/(2r_2).$$

If $p<0$ and $\zeta\geqslant 0$, the minimum is located at the p axis. Therefore

$$p=0 \quad \text{and} \quad \zeta=\max[0,-\vec{q}\cdot\vec{\nabla}h-\mu_1/(2r_2)].$$

If $p\geqslant 0$ and $\zeta<0$, the minimum is located at the ζ axis. Therefore

$$p=\max[0,\tfrac{1}{2}[|P|-\gamma(1-\mu_1+2r_2\vec{q}\cdot\vec{\nabla}h)]/(r_2\gamma^2+2r_3) \quad \text{and} \quad \zeta=0.$$

If $p<0$ and $\zeta<0$ the minimum can be either at the p axis or at the ζ axis:

$$p=\tfrac{1}{2}\{|P|-\gamma[1+2r_2(\vec{q}\cdot\vec{\nabla}h-\mu_1)]\}/(r_2\gamma^2+2r_3) \quad \text{and} \quad \zeta=0 \quad \text{(A3)}$$

or

$$p=0 \quad \text{and} \quad \zeta=-\vec{q}\cdot\vec{\nabla}h-\mu_1/(2r_2). \quad \text{(A4)}$$

If Eqs. (A3) and (A4) do not provide non-negative values of p and ζ, then

$$p=0 \quad \text{and} \quad \zeta=0.$$

All calculations were performed on a uniform rectangular 60×60 mesh by a finite-difference method on a simple five-point stencil.

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